QUANTUM FIELD THEORY AND THE FREE SCALAR FIELD

The first example of a quantum field theory is the quantised Klein-Gordon field. The common heuristic is that we have placed a harmonic oscillator at every point in space. Often what is minimally desired of quantization is that it provides a 'Hilbert space' on which our observables A in the classical theory become Hermitian operators $\hat{A}$ and so that

$$\frac{\imath}{\hbar}\widehat{\{A, B\}} = [\hat{A}, \hat{B}]$$

This is commonly called canonical quantization. From here on, our notation will not distinguish between the classical observable and its quantised version. As in quantum mechanics we have a Hamiltonian operator H that is reponsible for the time evolution of the system. We also want any operator we consider to have a complete set of eigenvectors. The position ϕ and momentum π observables satisfy the following equal time commutation relations

$$\{\phi(t, x), \pi(t', x')\} = \delta(x - x')$$

$$\{\phi(t, x), \phi(t, x')\} = \{\pi(t, x), \pi(t, x')\} = 0$$

Recall that we know this is true at time zero.

Exercise Show that the equal time commutation relations hold for nonzero time. If you so desire, find the nonequal time commuation relations.

For the Klein-Gordon system, the Hamiltonian is

$$H = \frac{1}{2}\int (\pi^2 + |\nabla\phi|^2 + m^2\phi^2)dx$$

Assume we are in units in which $\hbar = 1$, then we get the equal time commutation relations

$$[\phi(t, x), \pi(t, x')] = \imath\delta(x - x')$$

$$[\phi(t, x), \phi(t, x')] = [\pi(t, x), \pi(t, x')] = 0$$

Notice that something is strange already. If ϕ and π were both represented by operator-valued functions, their commutator would be a function also. We will come back to this point in a bit, but for now we proceed as if everything is hunky-doory. Using the Fourier transform, we can rewrite solutions to the Klein-Gordon equation as a superposition of forward and backward moving waves

$$\phi(t, x) = \int \frac{dk}{\sqrt{(2\pi)^n 2\omega_k}} \left(a(k)e^{-\imath(\omega_k t - kx)} + a^\dagger(k)e^{\imath(\omega_k t - kx)}\right)$$

where $\omega_k = \sqrt{k^2 + m^2}$ and $a^\dagger(k)$ is the complex conjugate of $a(k)$. This gives

$$\dot{\phi}(t, x) = \pi(t, x) = \int \frac{-dk\imath\omega_k}{\sqrt{(2\pi)^n 2\omega_k}} \left(a(k)e^{-\imath(\omega_k t - kx)} + a^\dagger(k)e^{\imath(\omega_k t - kx)}\right)$$

Using these two expression we can solve for a and $a^\dagger$ in terms of ϕ and π.

$$a^\dagger(k) = \int dx e^{-\imath(\omega_k t - kx)}(\omega_k\phi(x) - \imath\pi)$$

$$a(k) = \int dx e^{\imath(\omega_k t - kx)}(\omega_k \phi + \imath \pi)$$

Now if we quantize we get the commutation relations

$$[a(k), a^\dagger(k')] = \delta(k - k')$$

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0$$

Note the similarities to the quantum harmonic oscillator. The quantized Hamiltonian can be reexpressed in terms of the creation and annihilation operators

$$H = \frac{1}{2}\int dk \omega_k (a^\dagger a + a a^\dagger)$$

We can then use the commutation relations to get

$$H = \frac{1}{2}\int dk (2\omega_k a^\dagger a + \delta(0))$$

Quantum field theory is famous for its many infinties that need to be argued away. Since the quantised Klein-Gordon system is analogous to placing a quantum harmonic oscillator at each point in space and summing (or integrating). As we saw in a previous lecture, the harmonic oscillator has positive vacuum energy. Hence, any total vacuum energy of any infinite sum of harmonic oscillators will diverge. To modify the Hamiltonian, we formally subtract the 0-point energy from H and let our new Hamiltonian be

$$H = \int dk \omega_k a^\dagger a$$

One checks that H generates the same quantum equations of motion - which are really all that matter.

Exercise Double check the calculations above. Also, check that our modified Hamiltonian has the correct equations of motion.

In analogy to the quantum harmonic oscillator, we can define the number operator

$$N = \int dk a^\dagger a$$

and the momenta operators

$$P^\mu = \int dk k^\mu a^\dagger a$$

Exercise Check that $a^\dagger$ is really a creation operator and a is an annihilation operator, i.e. check that $a^\dagger(k)|n\rangle = |n+1\rangle$ and $a(k)|n\rangle = |n-1\rangle$ where $|n\rangle$ is eigenvector of N with eigenvalue n. Also determine the similar relations when $a^\dagger, a$ acts on eigenvectors of P^μ.

These operators commute with H and hence yield conserved quantities, namely the total number of particles and the total momentum. If we assume we have a vacuum $|0\rangle$, then the vector

$$a^\dagger(k_1)\cdots a^\dagger(k_n)|0\rangle$$

is a system of n particles (eigenvector of the number operator with eigenvalue n) with ith particle have momentum k_i. The Fock space of the quantised Klein-Gordon system can be thought of as the 'Hilbert space' generated by expressions of the form above. We now return to our senses and realize that what we have been treating as operator-valued functions on $\mathbb{R}^n$ must be actually be operator-valued distributions.

To get actual operators we need to pair our operator-valued distributions with a test function $f(x)$. Keeping this in mind, we give a rigorous construction of the Fock space associated the quantised Klein-Gordon equation. Objects of our Hilbert space $\mathcal{H}$ are going to be infinite sequences of symmetric complex-valued functions on $\mathbb{R}^n$

$$\{F_0, F_1(k_1), F_2(k_1, k_2), F_3(k_1, k_2, k_3), \cdots\}$$

which have finite norms with respect to the Hermitian inner product

$$(G, F) = \sum_{n \geq 0} \frac{1}{n!} \int F_n \overline{G}_n dk_1 \cdots dk_n$$

The creation and annihilation operators are operator-valued distributions. For each test function $f(k)$ we set

$$(a[f]F)_n(k_1, \ldots, k_{n-1}) = \int f(k) F_n(k_1, \ldots, k_{n-1}, k) dk$$

and

$$(a^\dagger[f]F)_n(k_1, \ldots, k_{n+1}) = \sum_{i=1}^{n} f(k_i) F_n(k_1, \ldots, \hat{k_i}, \ldots, k_{n+1})$$

For more information see [2].

References

[1] Greiner, Walter and Joachim Reinhardt. *Field Quantization.*
[2] Rabin, Jeffrey M. "Introduction to Quantum Field Theory for Mathematicians." *Geometry and Quantum Field Theory.*