OUTLINE OF CLASSICAL FIELD THEORY

The first natural question is: what is a field? A good definition that encompasses the main cases in physical theories is that a field is a section of a smooth fiber bundle $P \to M$.

Examples

(1) If we take $P = M \times X$, then sections are the same as smooth maps $M \to X$. The types of field theories are called sigma models.
(2) Often P is a vector bundle. Sections of $TM^{\otimes r} \otimes T^*M^{\otimes s}$ are tensor fields.
(3) Connections on a fixed vector bundle P are another common type of field. The difference of two connections lies in $End(P) \otimes T^*M$.
(4) Studying gravity takes metrics which are sections of $Sym^2(T^*M)$.

As before, we will deal with the most simple situation, that of maps from $\mathbb{R}^{n+1}$ to $\mathbb{R}$. There are two ways to view fields in the simple setting. One, we could take our space of fields to be the Frechet space $\mathcal{F}$ of compactly support smooth maps from $\mathbb{R}^n$ to $\mathbb{R}$. The action then is a functional on the path space of our space of fields. Or, for any path $\gamma : \mathbb{R} \to \mathcal{F}$, we can define a function $\tilde{\gamma} : \mathbb{R}^{n+1} \to \mathbb{R}$ by $\tilde{\gamma}(t, x) = \gamma(t)(x)$. In classical mechanics we extremized paths in a finite dimensional manifold; in classical field theory, we extremize paths in infinite dimensional manifolds. A common heuristic is that a field $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuum limit of a large number of particles. If the index i labels the particle, we think that $q(i, t) \to \phi(x, t)$ as the number of particles becomes infinite. We have placed a classical mechanical system at each point in space. Our equations of motions (derived from physical considerations or experimental results) usually come in the form of a partial differential equation F and we look for solutions

$$F\phi = 0$$

As before, we seek an action whose Euler-Lagrange equation gives the PDE. Usually, the Lagrangian is of the following form

$$S(\phi) = \int_{\mathbb{R}^n} L(\phi, \nabla\phi, x) dx dt$$

where

$$L : \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$$

is smooth. The Euler-Lagrange equations are then

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\phi + \varepsilon\psi) = \int_{\mathbb{R}^n} \left(\frac{\partial L}{\partial \phi}\psi + \frac{\partial L}{\partial(\partial_\mu \phi}\partial_\mu \psi \right) dx$$

We assume that ψ vanishes at infinity to perform integration by parts to get

$$\left(\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)} \right) \psi = 0$$

Example Our main is the free scalar field given by the Klein-Gordon equation (i.e. the wave equation).

$$(\Box + m^2)\phi = 0$$

where $\Box$ is Laplacian of $\mathbb{R}^{n+1}$ with the Minkowski metric g, i.e.

$$g((t,x),(t',x')) = tt' - x \cdot x'$$

where $\cdot$ is the usual dot product. Explicitly

$$\Box = \partial_t^2 - \partial_{x^1}^2 - \cdots - \partial_{x^{n-1}}^2$$

One can take two perfectly good actions related by integration by parts.

$$S = -\frac{1}{2}\int \phi(\Box + m^2)\phi dxdt$$

or

$$S = \frac{1}{2}\int (\partial^\mu \phi \partial_\mu \phi - m^2\phi^2) dxdt$$

The reader can easily check that the Euler-Lagrange equation for these actions is Klein-Gordon equation.

As mentioned earlier, there is a stronger more local version of Noether's theorem in field theory. One can take infinitesimal variations of the field and/or the coordinates. Here, we will work out the case where only the field varies and state the answer in the general case. We will adopt more physics notation and denote the variation by $\phi \to \phi + \delta\phi$.

A quick aside to explain said notation: Here $\phi \to \phi + \delta\phi$ means we have (at least) a one parameter group action on the space of fields. The term $\delta\phi$ should be thought of as the linearization of the group action at ϕ. In the finite dimensional setting, when one linearizes a one parameter group of diffeomorphisms, one gets a vector field that is the infinitesimal generator of the action. In the case of Banach and Frechet manifolds, one should have faith that everything works similarly. (If your faith is lacking, try reading [2] or [3] for calculus on Frechet manifolds or [4] for Banach manifolds). Here $\delta\phi$ is the value of the vector field at ϕ. Heuristically, a vector field on the space of fields should be, for each field ϕ, a choice of vector field along the image of ϕ, i.e. a section of ϕ^*TP. In the simple case where our space of fields is the set of smooth maps $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$, vector fields along the image of ϕ can be identified with a family of smooth functions $X_\phi : \mathbb{R}^n \to \mathbb{R}$ depending on the field ϕ. Comparing the notation, we have $\delta\phi = X_\phi$.

Assuming the action is preserved we have.

$$\delta S = 0 = \int \left(\frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi)\right) dxdt =$$

$$\int \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu\phi)}\delta\phi\right) dxdt + \int \left(\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)}\right)\delta\phi dxdt$$

If S is in fact invariant over all domains $V \subset \mathbb{R}^n$ and using the Euler-Lagrange equations, we must have

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu\phi)}\delta\phi\right) = 0$$

Let us set

$$j^\mu_\phi = \frac{\partial L}{\partial(\partial_\mu \phi)}\delta\phi$$

j is called the Noether charge. Let

$$Q_{V,\phi}(t) = \int_V j^0_\phi(t,x)dx$$

Stokes' theorem tells us that the flux out of the boundary V of the spatial component of $-j_\phi$ equals the rate of change of $Q_{V,\phi}$ in time. In particular, if we take $V = \mathbb{R}^{n+1}$ and assume $\delta\phi$ vanishes at infinity, we see that $Q_{\mathbb{R}^{n+1},\phi} = Q_\phi$ is independent of t, i.e. is a conserved quantity. Q the Noether charge for the symmetry $\delta\phi$. Notice that the Noether charge is a operator on the space fields (we have a freedom to choose any point ϕ in the space of fields to look at the infinitesimal symmetry at). It is only preserved, in time evolution, if we are choose a field that is an extremal of S.

In the case that we take a variation of the field $\phi \to \delta\phi$ and a variation of the coordinates $x^i \to \delta x^i$, we have that the Noether current is

$$j^\mu_\phi = (\delta\phi - \delta x^\nu \partial_\nu \phi)\frac{\partial L}{\partial(\partial_\mu \phi)} + L\delta x^\mu$$

For a derivation see [1]. Again all of this can be formulated in a manifestly invariant manner and generalized to manifolds with metrics of Lorentzian signature, for more information see the notes by Deligne and Freed, [5].

We now we give the Hamiltonian formulation of classical field theory. We proceed in complete analogy with Hamiltonian mechanics. The conjugate momentum $\pi(t, x)$ is given by

$$\pi(t, x) = \frac{\partial L}{\partial(\partial_t \phi)} = \frac{\partial L}{\partial \dot{\phi}}$$

One hopes that the equations of motion are nice enough so that the PDE

$$F\phi = 0$$
$$\phi(0, x) = \psi(x)$$
$$\partial_t \phi(0, x) = \sigma(x)$$

is uniquely solvable. To specify an extremal ϕ of the action, it is enough to specify $\phi(0, x)$ and $\partial_t\phi(0, x)$. The analogy with the case of classical mechanics is that this set of data should be viewed as living on the tangent bundle to space of compactly supported smooth maps from $\mathbb{R}^n$ to $\mathbb{R}$. When we solve for the conjugate momentum, we set $t = 0$ since that should be enough to determine the extremal. We are really viewing $\pi(x)$ as a covector via

$$\xi(x) \to \int \pi(x)\xi(x)dx$$

If I let $\mathcal{F}$ denote the space of compactly supported smooth maps as before, then $(\phi, \pi) \in T^*\mathcal{F} \cong \mathcal{F} \times \mathcal{F}$.

We again assume we can solve for $\partial_t\phi$ in terms of π and ϕ using the above equation. The Hamitonian is defined to be

$$H = \int (\pi\dot{\phi} - L)dx$$

The Hamiltonian will not be time dependent since it commutes with itself under the Poisson bracket (see below).

In this setting, Hamilton's equations of motions are usually described using the language of the functional derivatives. Start with our general definition of a space of fields, namely sections of some fiber bundle $P \to M$. Let F be a differentiable function on this space. The tangent space at a point ϕ is the space of sections of the pulled-back tangent bundle, $\Gamma(\phi^*TP)$. Inside, the tangent space we have the space of smooth compactly supported sections. dF_ϕ lies in the dual space so can be represented as a distribution which we will call the functional derivative $\frac{\delta F}{\delta \phi}$ of F at ϕ. If $\frac{\delta F}{\delta \phi}$ makes sense as a function, its value at a point x should be thought of as the component of the differential of F at ϕ corresponding to the differential of the coordinate function given by evaluating at x. Let us compute a few cases.

Examples

(1) If we take the functional given by evalution at a point x, its Frechet derivative can be identified with itself. The kernel of evaluation at a point x is the Dirac delta δ_x. Note that it is independent of ϕ.

(2) For a functional F given by intergrating against a kernel K a quick computation will convince you that the functional derivative at ϕ is the Euler-Lagrange expression at ϕ (not set equal to zero). In particular for an action of the type mentioned at the beginning of the notes, we have

$$\frac{\delta S}{\delta \phi(x)} = \frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)}$$

(3) If we have a direct product of spaces, like in our case $T^*\mathcal{F} \cong \mathcal{F} \times \mathcal{F}$, we can also take partial functional derivatives where we consider the linearization acting each component seperately.

In this language, Hamilton's equations of motion for a path in $T^*\mathcal{F}$,are

$$\frac{\delta H}{\delta \pi} = \dot{\phi} \quad \frac{\delta H}{\delta \phi} = -\dot{\pi}$$

or

$$dH_{(\phi,\pi)}(X) = \int (\dot{\phi} X_\pi - \dot{\pi} X_\phi) dx$$

Here X_ϕ represents the component of the vector field along the base and X_π is the component of the vector field along the fiber. We now compute to verify these equations are equivalent to

$$\frac{\delta S}{\delta \phi} = 0$$

We take the differntial

$$dH_{(\phi,\pi)}(X) = \int \left(X_\pi \dot{\phi} + \pi \dot{X}_\phi - \frac{\partial L}{\partial \phi} X_\phi - \frac{\partial L}{\partial(\partial_\mu \phi)} \partial_\mu (X_\phi) \right) dx$$

Using the definitions the term $\pi \dot{\delta\phi}$ cancels with the time derivative, we see that

$$\frac{\delta H}{\delta \pi} = \dot{\phi} \quad \frac{\delta S}{\delta \phi} = -\frac{\partial L}{\partial \phi} + \sum_{\mu=1}^{n-1} \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)}$$

This last part equals $-\dot{\pi}$ if and only if the Euler-Lagrange equations are satisfied.

In classical field theory, we also have a Poisson bracket coming from a natural symplectic structure. The symplectic form acts on the pair (X, Y) of vector fields with compact support to give

$$\int \int (X_\phi(x) Y_\pi(y) - X_\pi(y) Y_\phi(x)) dx dy$$

The Poisson bracket can be expressed in the standard way on an infinite dimensional symplectic manifold (even though the cotangent and tangent spaces might not be isomorphic). On infinite dimensional symplectic manifold $\mathcal{X}$, for each point $x \in \mathcal{X}$ we have a nondegenerate pairing

$$\omega_x : T_x\mathcal{X} \times T_x\mathcal{X} \to \mathbb{R}$$

which induces a nondegenerate pairing on the dual spaces and hence an embedding

$$\omega_x^* : (T_x\mathcal{X})^* \to (T_x\mathcal{X})^{**}$$

To define the Poisson bracket on smooth function $A, B : \mathcal{X} \to \mathbb{R}$ we set

$$\{A, B\}(x) = \omega_x^*(dA_x)(dB_x) - \omega_x^*(dB_x)(dA_x)$$

A morally correct definition often used by physicists is

$$\{A, B\}(\phi, \pi) = \int \left(\frac{\partial A}{\partial \phi} \frac{\partial B}{\partial \pi} - \frac{\partial B}{\partial \phi} \frac{\partial A}{\partial \pi} \right) dx$$

Exercise Show that the above formula is morally correct (i.e. determine the circumstances where it makes sense and equals the standard definition). Verify all the Poisson brackets below.

In the case of H

$$\dot{A} = \{A, H\}$$

On our space $T^*\mathcal{F}$ for each $x \in \mathbb{R}^n$ we have functions

$$\phi(x)(\psi, \tau) = \psi(x)$$
$$\pi(x)(\psi, \tau) = \pi(x)$$

We get the following commutation relations.

$$\{\phi(x), \pi(x')\} = \delta(x - x')$$
$$\{\phi(x), \phi(x')\} = \{\pi(x), \pi(x')\} = 0$$

Note they are the analog of the relations in classical mechanics.

$$\{q_j, p_i\} = \delta_{ij}$$
$$\{q_j, q_i\} = \{p_j, p_i\} = 0$$

This last observation hits home the central point of this treatment. We want to do the same things in classical field theory as we did in classical mechanics even though it may require some more mathematical sophistication. The closer the our descriptions of mechanics and field theory are; the closer the descriptions of their quantisations will be.

References

[1] Greiner, Walter and Joachim Reinhardt. *Field Quantization.*
[2] Hamilton, Richard. "The inverse function theorem of Nash and Moser." *Bulletin of the American Mathematical Society (N.S.)* 1982.
[3] Kriegl, Andreas and Peter Michor. *The Convenient Setting of Global Analysis*
[4] Lang, Serge. *Fundamentals of Differential Geometry*
[5] Deligne, Pierre et al. eds. *Quantum Fields and Strings: A Course for Mathematicians. Volume 1.*