OUTLINE OF CLASSICAL MECHANICS

Newton's laws of motion are succint and have proven quite impressive, but there are more useful formulations of them. The two most prominent being Lagrangian and Hamiltonian formulations. We will first review the transitions and relations between the three perspectives in Euclidean space. Then we will discuss the natural extension to manifolds.

We start with Lagrangian mechanics for a particle moving in $\mathbb{R}^n$. The Lagrangian density $L(q, v, t)$ is a function on $T\mathbb{R}^n \times \mathbb{R}$. We can consider the following functional on space of smooth paths in $\mathbb{R}^n$

$$q(t) \mapsto S(q(t)) = \int_{t_0}^{t_1} L(q(t), \dot{q}(t), t)dt$$

We assume that L is differentiable. Let us quickly compute the form of the Euler-Lagrange equations.

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_{t_0}^{t_1} L(q(t) + \varepsilon h(t), \dot{x} + \varepsilon h(t), t)dt = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} h(t) + \frac{\partial L}{\partial v} \dot{h}(t) \right) dt$$

We assume $h(t)$ vanishes at the boundary to allow integration by parts:

$$\int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} h(t) + \frac{\partial L}{\partial v} \dot{h}(t) \right) dt = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial v} \right) h(t)dt$$

So looking for an extremal for the action, we get the following Euler-Lagrange equations.

$$\frac{\partial L}{\partial q}(q, \dot{q}, t) - \frac{d}{dt} \frac{\partial L}{\partial v}(q, \dot{q}, t) = 0$$

Let us consider a useful example. If we take a particle of mass m moving under the influence of a potential $U(q)$ (we could take it to be time dependent if we desired), Newton's second law reads

$$m\ddot{q} + \frac{\partial U}{\partial q} = 0$$

The kinetic energy $T = (1/2)m\dot{q}^2$ and the potential energy is given by U. Consider the Lagrangian density given by $T - U$. Now if we compute the the Euler-Lagrange equations, we get

$$-\frac{\partial U}{\partial q}(q(t)) - m\ddot{q}(t) = 0$$

We see that the Newton's equations of motion arise as the Euler-Lagrange equations under the proper choice of Lagrangian density. This is commonly called Hamilton's principle of least action.

Symmetries play an essential role in every level of physics. They are not just for computational ease; symmetries are built into the foundations of many theories. There is an overall guiding meta-theorem in physics which goes roughly as follows.

Theorem (Noether's Theorem) Symmetries of the physical system lead to conserved quantities.

We will see a few formulations of Noether's theorem as this course progresses. Below we present a fairly simple mathematical version of Noether's theorem in the setting of classical mechanics, this version is taken from [1].

Theorem In our setting, we have a smooth map

$$L : T\mathbb{R}^n \to \mathbb{R}$$

which is our Lagrangian density. Assume we have a smooth one parameter family of smooth diffeomorphisms

$$h^s : \mathbb{R}^n \to \mathbb{R}^n$$

so that

$$L \circ h_*^s = L$$

Then,

$$I(q, v) = \frac{\partial L}{\partial v^i} \frac{dh^s}{ds}\bigg|_{s=0}$$

is a conserved quantity, i.e. it is constant on trajectories.

Proof Since $L \circ h^s = L$ if we differentiate by s we get

$$\frac{d}{ds}\bigg|_{s=0} L \circ h^s = \frac{\partial L}{\partial q^i} \frac{dh^s}{ds}\bigg|_{s=0} + \frac{\partial L}{\partial v^i} \frac{dh_*^s}{ds}\bigg|_{s=0} = 0$$

Here the lower star denotes the linearization of h^s, $h_*^s : T\mathbb{R}^n \to T\mathbb{R}^n$. If $x(t)$ is an extremum of the action S, it satisfies the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial q^i}$$

We substitute in

$$\frac{d}{ds}\bigg|_{s=0} L \circ h^s = \left(\frac{d}{dt} \frac{\partial L}{\partial v^i} \right) \frac{dh^s}{ds}\bigg|_{s=0} + \frac{\partial L}{\partial v^i} \left(\frac{d}{dt} \frac{dh^s}{ds}\bigg|_{s=0} \right) = \frac{d}{dt} \left(\frac{\partial L}{\partial v^i} \frac{dh^s}{ds}\bigg|_{s=0} \right) = 0$$

□

This generalizes in a straightforward way to the case where Euclidean space is replaced by a smooth manifold M. An invariant way to write the conserved quantity in this case is

$$I = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} L(x, v + \varepsilon \left(\frac{dh^s}{ds}\bigg|_{s=0} \right)_x)$$

One can see from this that we really only need to know about the invariance under the action of the vector field on M that generates the one parameter group of diffeomorphisms. (It is standard that invariance under a one parameter group of diffeomorphisms is equivalent to invariance with respect to the Lie derivative by the infinitesimal generator of the action, see [2]). Often the conserved quantity of an infinitesimal action are called its Noether charge. (The terminology comes from the fact that charge is the conserved quantity associated to a $U(1)$ symmetry of the electromagnetic action). Note that this is not the most general form of a symmetry that we could take. Instead of insisting the symmetry preserves the Lagrangian density, we could have only assumed that it preserves the action. We will do this when we come to field theory.

Why transition from Newton's law to Lagrangian formalism? Well, it is manifestly more invariant which allows immediate generalizations to manifolds. Also it is more generic and elegant (in my opinion). Noether charges can be directly obtained from symmetries of the system. One disadvantage is that the Lagrangian formalism does not make it simple to express the time evolution of the system. This disadvantage will be remedied by an alternative formulation.

To get the alternative formulation, we apply the Legendre transform to L. For the general definition and some more thorough discussion see [1]. Define

$$p_i = \frac{\partial L}{\partial v^i}$$

as the conjugate momentum to position q^i. Note that the p_i may depend on t. In most physical situations, the equations of motions is quadratic in the time derivatives. If we specify the position at time zero and the velocity vector at time zero, we know the trajectory uniquely. So we simply set t equal to zero in the p_i. We are really viewing $\mathbb{R}^n \times \mathbb{R}^n$ as the space of solutions to the equations of motion. We assume that we can invert the preceeding equation and solve for v_i in terms of the p_i and q^i. We then define

$$H(q,p,t) = p_i v^i - L(q,v,t)$$

We call this the Hamiltonian of the system. The Euler-Lagrange equations are converted into Hamilton's equations

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} \quad \dot{q}^i = \frac{\partial H}{\partial p_i}$$

To check this, we compute the exterior derivative of H

$$dH = v^i dp_i + p_i dv^i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial v^i} dv^i - \frac{\partial L}{\partial t} dt = v^i dp_i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial t} dt$$

Comparing with derivatives of H, we get

$$\frac{\partial H}{\partial p_i} = \dot{q}^i, \ \frac{\partial H}{\partial q^i} = -\frac{\partial L}{\partial q^i} = -\frac{d}{dt}\frac{\partial L}{\partial v^i} = -\dot{p}_i$$

Let us head back to the our only physical example. Our Lagrangian is

$$L(q,v) = \frac{1}{2}mv^2 - U(q), \text{so } p = \frac{\partial L}{\partial v} = mv$$

The Hamiltonian is then

$$H(q,p) = pv - L(q,v) = \frac{p^2}{2m} + U(q)$$

In a more general setting, if we have a Lagrangian of the form

$$L(q,v,t) = \sum a_{ij}(q,t) v_i v_j - U(q,t) = T - U$$

where we view U as the potential energy and $T = \sum a_{ij}(q,t) v^i v^j$ as the kinetic energy, the Hamiltonian is then of the form $H = T + U$. For a proof, see [1].

Now if we are given an observable $A(p,q)$, we can express its time evolution in the language of Poisson brackets.

$$\frac{d}{dt}A = \frac{\partial A}{\partial p_i}\dot{p}_i + \frac{\partial A}{\partial q^i}\dot{q}^i + \frac{\partial H}{\partial t} = -\frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q^i} + \frac{\partial A}{\partial q^i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial t} = \{A,H\} + \frac{\partial H}{\partial t}$$

where

$$\{A,B\} = -\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i} + \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i}$$

is the Poisson bracket.

This expression will become quite important when we quantize classical mechanics.

Just as in the Lagrangian setup, Hamiltonian mechanics works equally well if we replace $\mathbb{R}^n$ by a smooth manifold M. The conjugate momenta are viewed invariantly as components of a covector so the natural setting for Hamiltonian mechanics is the cotangent bundle of M, T^*M, just as the natural setting for Lagrangian mechanics is the tangent bundle TM. In fact much of the story persists if instead of taking T^*M, we take an arbitrary symplectic manifold. I do not know of a similar mathematical generalization of Lagrangian mechanics although TM does admit a natural integrable complex structure.

References

[1] Arnold, V.I. *Mathematical Methods of Classical Mechanics.*
[2] Lee, John. *Introduction to Smooth Manifolds.*