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Math 309: Quiz 1

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1. Solve the following boundary value problem for a function $u(x,t)$ for $0 \le x \le \pi$ and $0 \le t$:

$$\begin{aligned} u_{tt} &= u_{xx};\ 0 < x < \pi, 0 < t \\ u(0,t) &= 0;\ 0 < t \\ u(\pi,t) &= 0;\ 0 < t \end{aligned}$$

$$u(x,0) = \begin{cases} x, & 0 < x \le \pi/2 \\ \pi - x, & \pi/2 \le x < \pi. \end{cases}$$

Note that

$$\int_0^{\pi/2} x\sin(nx)\ dx = \frac{2\sin\left(\frac{n\pi}{2}\right) - \pi n\cos\left(\frac{n\pi}{2}\right)}{2n^2},$$

and

$$\int_{\pi/2}^{\pi} (\pi - x)\sin(nx)\ dx = \frac{2\sin\left(\frac{n\pi}{2}\right) + \pi n\cos\left(\frac{n\pi}{2}\right)}{2n^2}.$$

Hint: Start by looking for product solutions of the form

$$u(x,t) = X(x)\cos(\omega t).$$

We begin by considering product solutions of the form $X(x)\cos(\omega t)$, as suggested by the hint. The differential equation $u_{tt} = u_{xx}$ becomes

$$-\omega^2 X(x)\cos(\omega t) = X''(x)\cos(\omega t).$$

Canceling cosines gives

$$-\omega^2 X(x) = X''(x),$$

which has the general solution

$$X(x) = c_1\cos(\omega x) + c_2\sin(\omega x).$$

By the condition $u(0,t) = X(0)\cos(\omega t) = 0$ we can deduce that $X(0) = 0$ and similarly by $u(\pi,t) = 0$ we can deduce that $X(\pi) = 0$. The first condition guarantees that $X(0) = c_1 = 0$, and the second guarantees that $X(\pi) = c_2\sin(\omega\pi) = 0$, which implies that $\omega\pi = n\pi$ for some integer n, i.e. $\omega = n$. Thus our product solutions are (up to constants) of the form

$$\sin(nx)\cos(nt).$$

Since the conditions we have used so far are homogeneous, we can take linear combinations, so suppose

$$u(x,t) = \sum_{n=0}^{\infty} c_n \sin(nx)\cos(nt).$$

Plugging in the initial value condition, we see that

$$u(x,0) = \sum_{n=0}^{\infty} c_n \sin(nx) = \begin{cases} x, & 0 < x \le \pi/2 \\ \pi - x, & \pi/2 \le x < \pi. \end{cases}$$

Thus by Fourier analysis (taking a sine series for the rightmost function), we can conclude that

$$\begin{aligned} c_n &= \frac{2}{\pi}\int_0^{\pi} u(x,0)\sin(nx)\ dx \\ &= \frac{2}{\pi}\int_0^{\pi/2} x\sin(nx)\ dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin(nx)\ dx \\ &= \frac{2}{\pi}\left(\frac{2\sin\left(\frac{n\pi}{2}\right) - \pi n\cos\left(\frac{n\pi}{2}\right)}{2n^2}\right) + \frac{2}{\pi}\left(\frac{2\sin\left(\frac{n\pi}{2}\right) + \pi n\cos\left(\frac{n\pi}{2}\right)}{2n^2}\right) \\ &= \frac{2}{\pi}\left(\frac{4\sin(\frac{n\pi}{2})}{2n^2}\right) \\ &= \frac{4\sin(\frac{n\pi}{2})}{\pi n^2} \\ &= \begin{cases} 0, & n \text{ even} \\ \frac{4(-1)^{(n-1)/2}}{\pi n^2}, & n \text{ odd} \end{cases} \end{aligned}$$

Therefore the final solution can be written as

$$u(x,t) = \sum_{n=0}^{\infty} \frac{4(-1)^n}{\pi(2n+1)^2}\sin((2n+1)x)\cos((2n+1)t).$$