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Math 309 Practice Final 2

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Instructions: You may use a calculator for this exam. Please turn off all cell phones and pagers. You must show all work. Wherever a general solution is required, the solution must be in explicit form.

1. (25 points) Compute a sine Fourier series for the function

$$f(x) = \begin{cases} x, & 0 \le x \le 1/2 \\ 1 - x, & 1/2 \le x \le 1. \end{cases}$$

This is $\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$, where $L = 1$ and

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= 2\int_0^{\frac{1}{2}} x\sin(n\pi x)\,dx + 2\int_{\frac{1}{2}}^1 (1-x)\sin(n\pi x)\,dx \\ &= 2\int_{\frac{1}{2}}^1 \sin(n\pi x)\,dx + 2\int_0^{\frac{1}{2}} x\sin(n\pi x)\,dx - 2\int_{\frac{1}{2}}^1 x\sin(n\pi x)\,dx. \end{aligned}$$

We have

$$2\int_{\frac{1}{2}}^1 \sin(n\pi x)\,dx = -\frac{2}{n\pi}\cos(n\pi x)\Big|_{\frac{1}{2}}^1 = -\frac{2}{n\pi}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right),$$

and integrating by parts, we see that

$$\int_a^b x\sin(n\pi x)\,dx = \left[-\frac{x}{n\pi}\cos(n\pi x) + \frac{1}{(n\pi)^2}\sin(n\pi x)\right]_a^b$$

which gives

$$\begin{aligned}
2\int_0^{\frac{1}{2}} x\sin(n\pi x)\ dx - 2\int_{\frac{1}{2}}^1 x\sin(n\pi x)\ dx &= 2\left[-\frac{x}{n\pi}\cos(n\pi x) + \frac{1}{(n\pi)^2}\sin(n\pi x)\right]_0^{\frac{1}{2}} \\
&\quad - 2\left[-\frac{x}{n\pi}\cos(n\pi x) + \frac{1}{(n\pi)^2}\sin(n\pi x)\right]_{\frac{1}{2}}^1 \\
&= -\frac{1}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) \\
&\quad + \frac{2}{n\pi}\cos(n\pi) - \frac{1}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) \\
&= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}(-1)^n \\
&= \begin{cases} -\frac{2}{n\pi}(-1)^{n/2} + \frac{2}{n\pi} & n \text{ even} \\ \frac{4}{(n\pi)^2}(-1)^{(n-1)/2} - \frac{2}{n\pi} & n \text{ odd} \end{cases}
\end{aligned}$$

Thus

$$\begin{aligned}
a_n &= -\frac{2}{n\pi}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right) + \begin{cases} -\frac{2}{n\pi}(-1)^{n/2} + \frac{2}{n\pi} & n \text{ even} \\ \frac{4}{(n\pi)^2}(-1)^{(n-1)/2} - \frac{2}{n\pi} & n \text{ odd} \end{cases} \\
&= \begin{cases} 0 & n \text{ even} \\ \frac{4}{(n\pi)^2}(-1)^{(n-1)/2} & n \text{ odd} \end{cases}
\end{aligned}$$

2. (25 points) A one meter long elastic string is fixed at each end and its position $u(x,t)$ satisfies the PDE

$$u_{tt} = u_{xx}.$$

The string is plucked so that its initial position is

$$u(x,0) = \begin{cases} x, & 0 \le x \le 1/2 \\ 1-x, & 1/2 \le x \le 1. \end{cases}$$

Noting that this is the function from problem 1, solve for the function $u(x,t)$ with $0 \le x \le 1$ and $t \ge 0$ in terms of the Fourier series computed in problem 1.

Product solutions $X(x)T(t)$ must satisfy (since $u_{tt} = u_{xx}$)

$$\frac{X''}{X} = \frac{T''}{T} = \lambda$$

for some constant λ. Since each end of the string is fixed, we have $X(0) = X(1) = 0$. Thus, up to multiplication by a constant, $X(x) = \sin\left(\sqrt{-\lambda}x\right)$ from $X(0) = 0$, and since $X(1) = 0$, we have $\sin\left(\sqrt{-\lambda}\right) = 0$, which implies that $\sqrt{-\lambda} = n\pi$ for some integer n, i.e. $\lambda = -(n\pi)^2$.

Since this is the wave equation, we can assume that $T(t) = \cos(n\pi t)$, hence we obtain product solutions of the form $\sin(n\pi x)\cos(n\pi t)$. Thus our more general solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\cos(n\pi t).$$

Plugging in $t = 0$, we have that $u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$. Recognizing the given function for $u(x,0)$ from the previous problem, we conclude that these are the same a_n.

Hence

$$u(x,t) = \sum_{n=1,3,\dots}^{\infty} \frac{4}{(n\pi)^2}(-1)^{(n-1)/2} \sin(n\pi x)\cos(n\pi t).$$

3. (20 points) Find the general solution to the system of differential equations:

$$\mathbf{x}' = \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \sin t \\ 0 \end{pmatrix}.$$

Since $\text{charpoly}(A) = \lambda(\lambda - 2) + 2 = (\lambda - 1)^2 + 1$, the eigenvalues are $\lambda = 1 \pm i$. The corresponding eigenvectors are $\xi_{1+i} = \begin{pmatrix} -2 \\ 1+i \end{pmatrix}$ and $\xi_{1-i} = \begin{pmatrix} -2 \\ 1-i \end{pmatrix}$.

Recall that complex eigenvalues always come in conjugate pairs, and the corresponding eigenvectors are also conjugate pairs. If $A\xi = \lambda\xi$ and λ is complex with nonzero imaginary part, then with $\xi = \text{Re}(\xi) + i\text{Im}(\xi)$ and $\lambda = \text{Re}(\lambda) + i\text{Im}(\lambda)$, the part of the general (real) solution corresponding to the pair is

$$c_1[\text{Re}(\xi)\cos(\text{Im}(\lambda)t) - \text{Im}(\xi)\sin(\text{Im}(\lambda)t)]e^{\text{Re}(\lambda)t} + c_2[\text{Im}(\xi)\cos(\text{Im}(\lambda)t) + \text{Re}(\xi)\sin(\text{Im}(\lambda)t)]e^{\text{Re}(\lambda)t}.$$

In this case, this is the full general solution to the homogeneous part of the equation,

$$\begin{aligned}\mathbf{x}_h(t) &= c_1\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}\cos(t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(t)\right)e^t + c_2\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}\sin(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(t)\right)\sin(t)e^t \\ &= \begin{pmatrix} -2c_1 \\ c_1 + c_2 \end{pmatrix}\cos(t)e^t + \begin{pmatrix} -2c_2 \\ c_2 - c_1 \end{pmatrix}\sin(t)e^t.\end{aligned}$$

Using the method of undetermined coefficients, suppose a particular solution is $\mathbf{x}(t) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\sin(t)$. Then $\mathbf{x}'(t) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\cos(t)$, so this is not enough. We then guess that

$$\mathbf{x}(t) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\sin(t) + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}\cos(t) \Rightarrow \mathbf{x}'(t) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\cos(t) - \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}\sin(t).$$

Since

$$\begin{aligned}\begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}\mathbf{x}(t) + \begin{pmatrix} \sin t \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}\left[\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\sin(t) + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}\cos(t)\right] + \begin{pmatrix} \sin t \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2b_2 + 1 \\ b_1 + 2b_2 \end{pmatrix}\sin(t) + \begin{pmatrix} -2d_2 \\ d_1 + 2d_2 \end{pmatrix}\cos(t),\end{aligned}$$

we have that

$$\begin{aligned} b_1 &= -2d_2 \\ b_2 &= d_1 + 2d_2 \\ -d_1 &= -2b_2 + 1 \\ -d_2 &= b_1 + 2b_2. \end{aligned}$$

Then $-d_1 = -2b_2 + 1 = -2d_1 - 4d_2 + 1$ implies $d_1 = -4d_2 + 1$, and we have

$$d_2 = -b_1 - 2b_2 = 2d_2 - 2d_1 - 4d_2 = 2d_2 + 8d_2 - 2 - 4d_2 = 6d_2 - 2,$$

thus $d_2 = 2/5$, $d_1 = -4(2/5) + 1 = -3/5$, $b_2 = -3/5 + 2(2/5) = 1/5$ and $b_1 = -4/5$. Thus the general solution is

$$\mathbf{x}(t) = \begin{pmatrix} -2c_1 \\ c_1 + c_2 \end{pmatrix} \cos(t)e^t + \begin{pmatrix} -2c_2 \\ c_2 - c_1 \end{pmatrix} \sin(t)e^t + \begin{pmatrix} -4/5 \\ 1/5 \end{pmatrix} \sin(t) + \begin{pmatrix} -3/5 \\ 2/5 \end{pmatrix} \cos(t)$$

4. (20 points) With

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix},$$

compute e^A. You may express e^A as a product of matrices and their inverses. In other words, don't actually compute the inverse of M, just write M^{-1}.

First note that the eigenvalues of this upper triangular matrix are 1, 4, and 6. Corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 8/5 \\ 5/2 \\ 1 \end{pmatrix}$ respectively.

Thus with

$$T = \begin{pmatrix} 1 & 2/3 & 8/5 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{pmatrix},$$

we have that $T^{-1}AT = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$.

Thus

$$\begin{aligned} e^A &= e^{TDT^{-1}} = Te^DT^{-1} \\ &= \begin{pmatrix} 1 & 2/3 & 8/5 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ 0 & e^4 & 0 \\ 0 & 0 & e^6 \end{pmatrix} \begin{pmatrix} 1 & 2/3 & 8/5 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{pmatrix}^{-1}. \end{aligned}$$

5. (20 points) Solve the following initial value problem:

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

The matrix has one eigenvalue 2 of algebraic multiplicity 2 and geometric multiplicity 1. An eigenvector for 2 is $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and a generalized eigenvector (an η such that $(\lambda I - A)\eta = \xi$ is $\eta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The general solution is

$$c_1 \xi e^{2t} + c_2(\xi t + \eta)e^{2t} = \begin{pmatrix} (c_1 + c_2 t) \\ c_2 \end{pmatrix} e^{2t}.$$

When $t = 0$ we get

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

thus the solution is

$$\begin{pmatrix} (3 + 4t) \\ 4 \end{pmatrix} e^{2t}.$$