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Math 309 Practice Final

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Instructions: You may use a calculator for this exam. Please turn off all cell phones and pagers. You must show all work. Wherever a general solution is required, the solution must be in explicit form.

1. Find the general solution to the following first order system of equations:

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ e^{-t} \\ 0 \end{pmatrix}.$$

The homogeneous equation

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x}$$

is solved as follows. First, we note that since the matrix is upper triangular, its diagonal entries are its eigenvalues, i.e. 2 and 3. Note that 2 has multiplicity 2.

For $\lambda = 2$, the matrix $\lambda I - A$ is

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since this matrix has rank 2, its kernel has dimension 1, by the rank-nullity theorem. Therefore there is only one independent eigenvector $\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Since the algebraic multiplicity is 2, we must find a generalized eigenvector η such that $(\lambda I - A)\eta = \xi_1$. Thus we can use $\eta = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$. The part of the general solution corresponding to $\lambda = 2$ is therefore $c_1\xi_1 e^{2t} + c_2 (\xi_1 t + \eta) e^{2t}$.

For $\lambda = 3$, it is easily seen that $\xi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector, and the part of the general solution corresponding to $\lambda = 3$ is therefore $c_3\xi_2 e^{3t}$.

Thus the general solution to the homogeneous equation is

$$\mathbf{x}_h(t) = c_1\xi_1 e^{2t} + c_2 (\xi_1 t + \eta) e^{2t} + c_3 \xi_2 e^{3t},$$

and it only remains to find a particular solution to the system:

$$\begin{aligned} x_1' &= 2x_1 + x_2 + e^t \\ x_2' &= 2x_2 + e^{-t} \\ x_3' &= 3x_3 \end{aligned}$$

Note that $x_3(t) = 0$ is a solution. Using undetermined coefficients, suppose $x_2 = ce^{-t}$. Plugging in to $x_2' = 2x_2 + e^{-t}$, we obtain $-ce^{-t} = (2c+1)e^{-t}$, hence $-c = 2c + 1$, i.e. $c = -1/3$. So $x_2(t) = -\frac{1}{3}e^{-t}$. Plugging in to $x_1' = 2x_1 + x_2 + e^t$, we obtain $x_1' = 2x_1 - \frac{1}{3}e^{-t} + e^t$. Again using undetermined coefficients, suppose $x_1 = ae^{-t} + be^t$. Plugging in again gives

$$-ae^{-t} + be^t = 2(ae^{-t} + be^t) - \frac{1}{3}e^{-t} + e^t = (2a - \frac{1}{3})e^{-t} + (2b+1)e^t.$$

Thus $-a = 2a - \frac{1}{3}$, i.e. $a = \frac{1}{9}$ and $b = 2b + 1$, i.e. $b = -1$. Thus a particular solution is

$$\mathbf{x}_p(t) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{9}e^{-t} - e^t \\ -\frac{1}{3}e^{-t} \\ 0 \end{pmatrix}.$$

Finally, the general solution is

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t) = c_1\xi_1 e^{2t} + c_2\left(\xi_1 t + \eta\right)e^{2t} + c_3\xi_2 e^{3t} + \begin{pmatrix} \frac{1}{9}e^{-t} - e^t \\ -\frac{1}{3}e^{-t} \\ 0 \end{pmatrix}.$$

2. With

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 4 & 3 \end{pmatrix},$$

compute e^A as a product of explicit matrices by solving $\mathbf{x}' = A\mathbf{x}$ for a set of fundamental solutions $\{\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)\}$ such that

$$\begin{pmatrix} \mathbf{x}_1(0) & \mathbf{x}_2(0) & \mathbf{x}_3(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is an application of the uniqueness theorem, which asserts that a solution to an initial value problem is unique (note that e^{At} satisfies $\mathbf{x}' = A\mathbf{x}$). Since $e^{At} = I$ when $t = 0$, this implies that

$$e^{At} = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \mathbf{x}_3(t) \end{pmatrix}.$$

To solve this DE, we need to find the eigenvectors and eigenvalues of the matrix A. The characteristic polynomial is found to be $x^3 - 6x^2 - x + 6 = (x-1)(x+1)(x-6)$. Since the eigenvalues are all distinct, this matrix is diagonalizable, which makes the problem substantially easier.

Note that $\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector for $\lambda = 1$, and that $\xi_{-1} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ is an eigenvector for $\lambda = -1$. To find an eigenvector for $\lambda = 6$, note that $\lambda I - A = \begin{pmatrix} 5 & -2 & -3 \\ 0 & 4 & -3 \\ 0 & -4 & 3 \end{pmatrix}$.

Then $\xi_6 = \begin{pmatrix} 18/5 \\ 3 \\ 4 \end{pmatrix}$ is an eigenvector for $\lambda = 6$.

Note that with $T = \begin{pmatrix} \xi_1 & \xi_{-1} & \xi_6 \end{pmatrix}$, we have $T^{-1}AT = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 6 \end{pmatrix}$. Thus we can use the identity $e^{TDT^{-1}} = Te^{D}T^{-1}$ as follows:

$$
\begin{aligned}
e^A &= e^{TDT^{-1}} \\
&= Te^DT^{-1} \\
&= T \begin{pmatrix} e & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^6 \end{pmatrix} T^{-1} \\
&= \begin{pmatrix} 1 & 1 & 18/5 \\ 0 & 2 & 3 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 18/5 \\ 0 & 2 & 3 \\ 0 & -2 & 4 \end{pmatrix}^{-1}
\end{aligned}
$$

3. Find the unique solution to the following initial value problem:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Again, since the matrix is upper triangular, the eigenvalues are 1, 3, and -1.

For $\lambda = -1$, we find $\xi_{-1} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$.

For $\lambda = 1$, we find $\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

For $\lambda = 3$, we find $\xi_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The general solution is then

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{3t}.$$

Plugging in the initial condition gives

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

This is seen to imply that $c_1 = 1, c_2 = 0$, and $c_3 = 2$, and the final solution is:

$$\mathbf{x}(t) = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} e^{-t} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{3t} = \begin{pmatrix} 0 \\ 2e^{3t} - e^{-t} \\ 2e^{-t} \end{pmatrix}.$$

4. Find a cosine series for $f(x) = \sin(x)$ of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \text{ for } 0 \le x \le \pi.$$

Hint:

$$\sin(u)\cos(v) = \frac{\sin(u+v) + \sin(u-v)}{2}.$$

With the above form of cosine series for f, we have that for $n \neq 1$,

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi \sin(x)\cos(nx)\ dx \\
&= \frac{1}{\pi}\int_0^\pi (\sin((n+1)x) + \sin((1-n)x))\ dx \\
&= -\frac{1}{\pi}\left[\frac{\cos((n+1)x)}{n+1} + \frac{\cos((1-n)x)}{1-n}\right]_0^\pi \\
&= -\frac{1}{\pi}\left[\frac{\cos((n+1)\pi) - 1}{n+1} + \frac{\cos((1-n)\pi) - 1}{1-n}\right] \\
&= -\frac{1}{\pi}\left[\frac{(-1)^{n+1} - 1}{n+1} + \frac{(-1)^{1-n} - 1}{1-n}\right] \\
&= \begin{cases} 0, & n \text{ odd} \\ \frac{1}{\pi}\left(\frac{2}{n+1} + \frac{2}{1-n}\right) = \frac{4}{\pi(1-n^2)}, & n \text{ even} \end{cases}
\end{aligned}$$

When $n = 1$, note that

$$\begin{aligned}
a_1 &= \frac{2}{\pi}\int_0^\pi \sin(x)\cos(x)\ dx \\
&= \frac{1}{\pi}\int_0^\pi \sin(2x)\ dx \\
&= 0.
\end{aligned}$$

Thus by the substitution $n \to 2n$, we have

$$\sin(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos(2nx), 0 \le x \le \pi.$$

5. Compute the Fourier series of the function $f(x)$ on the interval $[-\pi, \pi]$ defined by

$$f(x) = \begin{cases} x, & -\pi < x < \pi \\ 0, & x = \pm\pi \end{cases}$$

Since f is an odd function, all of the cosine coefficients will be zero, and on $[-\pi, \pi]$ we have $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$, where

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\ dx \\
&= \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx)\ dx && x \sin(nx) \text{ is even} \\
&= \frac{2}{\pi} \left[-\frac{x}{n} \cos(nx) \right]_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} -\frac{1}{n} \cos(nx)\ dx && \text{IBP: } u = x, dv = \sin(nx)\ dx \\
&= \frac{2}{\pi} \left[-\frac{\pi}{n} (-1)^n - 0 \right] \\
&= \frac{2}{n} (-1)^{n+1}.
\end{aligned}$$

6. Suppose the temperature of a rod of length 2π is represented by

$$u(x,t), -\pi \leq x \leq \pi, t \geq 0,$$

and that

- $u(x,0) = x, -\pi \leq x \leq \pi,$
- $u(0,t) = 0, t \geq 0,$
- $u(\pi,t) = 0, t \geq 0,$ and
- $u_t = u_{xx}, -\pi < x < \pi, t > 0.$

Find a series solution of this boundary value problem.

We begin by considering separable solutions to the boundary value problem of the form $X(x)T(t)$. Plugging in the condition that $u_t = u_{xx}$, we obtain that

$$X(x)T'(t) = X''(x)T(t),$$

hence

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Note that since these are functions of independent variables, they must both be equal to the same constant, call it λ, giving the pair of ordinary differential equations:

$$X''(x) = \lambda X(x); \quad T'(t) = \lambda T(t).$$

The condition that $u(0,t) = 0$ becomes $X(0)T(t) = 0$, whence we can assume that $X(0) = 0$. Similarly, we deduce that $X(\pi) = 0$ from $u(\pi,t) = 0$. The general solution to $X'' = \lambda X$ is

$$X(x) = c_1 \cos(\sqrt{-\lambda}x) + c_2 \sin(\sqrt{-\lambda}x).$$

$X(0) = 0$ implies that $0 = c_1$, and $X(\pi) = 0$ implies that $0 = c_2 \sin(\sqrt{-\lambda}\pi)$, which in turn means that $\sqrt{-\lambda}\pi = n\pi$ for some integer n, i.e. that $\lambda = -n^2$. Thus our solutions for X (up to constants) are

$$X_n(x) = \sin(nx).$$

Turning our attention to T, we are considering the equation $T' = -n^2T$, which has general solution $T_n(t) = ce^{-n^2t}$. Thus our product solutions are (up to constants):

$$\sin(nx)e^{-n^2t}.$$

All of the conditions we have used so far have been homogeneous, so we may freely take linear combinations, assuming now that our solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)e^{-n^2t},$$

for some constants c_n (we start summing at 1 since for $n = 0$, $\sin(nx) = 0$). Plugging in the initial value gives:

$$u(x,0) = x = \sum_{n=1}^{\infty} c_n \sin(nx),$$

which reveals that the c_n are the sine coefficients in the Fourier series of the previous problem, hence

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx) e^{-n^2 t}.$$