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Math 309D Midterm Solutions

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Instructions: You may use a calculator for this exam. Please turn off all cell phones and pagers. You must show all work. Wherever a general solution is required, the solution must be in explicit form.

1. Draw a phase portrait for the following system of differential equations:

$$\mathbf{x}' = \begin{pmatrix} 2 & 2 \\ -3 & 5 \end{pmatrix} \mathbf{x}$$

The idea here was to look at the characteristic polynomial of the matrix:

$$\chi_A(x) = \begin{vmatrix} x-2 & -2 \\ 3 & x-5 \end{vmatrix} = (x-2)(x-5) + 6 = x^2 - 7x + 16.$$

The roots are

$$\frac{7 \pm \sqrt{49 - 4 \cdot 16}}{2} = \frac{7 \pm \sqrt{-15}}{2},$$

which are complex conjugates with positive real part. Thus the phase portrait will look like an outward spiral. Computing $\mathbf{x}'$ at $(0, 1)$, which is

$$\begin{pmatrix} 2 & 2 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix},$$

reveals that this should be a clockwise spiral.

2. Find the general solution to the following system of differential equations.

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}.$$

First we note that the matrix is symmetric, thus it is diagonalizable.

$$\chi_A(x) = \begin{vmatrix} x-1 & -2 \\ -2 & x-1 \end{vmatrix} = (x-1)^2 - 4 = x^2 - 2x - 3 = (x-3)(x+1).$$

Thus the eigenvalues are -1 and 3, and one can verify that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are corresponding eigenvectors, respectively.

We form $T = \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and we have that $T^{-1}AT = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$. We make the change of variable $\mathbf{x} = T\mathbf{y}$ and multiply on the left by $T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$:

$$T^{-1}T\mathbf{y}' = T^{-1}AT\mathbf{y} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^t \\ 0 \end{pmatrix}.$$

Cleaning up:

$$\mathbf{y}' = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{y} + \frac{1}{2}\begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

This is:

$$\begin{aligned} y_1' &= -y_1 + \frac{e^t}{2} \\ y_2' &= 3y_2 + \frac{e^t}{2}. \end{aligned}$$

Thus the solutions are $y_1 = c_1 e^{-t} + \frac{1}{4}e^t$ and $y_2 = c_2 e^{3t} - \frac{1}{4}e^t$. Applying T gives

$$\mathbf{x} = T\mathbf{y} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_1 e^{-t} + \frac{1}{4}e^t \\ c_2 e^{3t} - \frac{1}{4}e^t \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} + \frac{1}{4}e^t + c_2 e^{3t} - \frac{1}{4}e^t \\ c_2 e^{3t} - \frac{1}{4}e^t - c_1 e^{-t} - \frac{1}{4}e^t \end{pmatrix}.$$

3. Find the general solution to the following system of differential equations.

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

By inspection, since the matrix is upper triangular, there is one eigenvalue of double multiplicity 1, and the eigenspace is dimension 1, spanned by $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus we must find a generalized eigenvector η which satisfies

$$(A - \lambda I)\eta = (A - I)\eta = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \xi,$$

and $\eta = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$ is a solution. Thus the general solution of the homogeneous equation is

$$\mathbf{x}_h = c_1 \xi e^t + c_2(\xi t + \eta)e^t.$$

If we make a lucky guess at the form of the particular solution, we guess $\mathbf{x}_p = \mathbf{c}(\cos t + \sin t)$, where $\mathbf{c}$ is a constant vector. Then $\mathbf{x}_p' = \mathbf{c}(\cos t - \sin t)$ and

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{c}(\cos t + \sin t) + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

One finds that $\mathbf{c} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$ solves the equality, and the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \right] e^t + \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} (\cos t + \sin t).$$

4. There are three 1000 liter (L) tanks of water in a factory. Water flows into tank 1 at a rate of 1 L/min from an outside source. Water flows from tank 1 to tank 2 at a rate of 2 L/min, and from tank 2 to tank 1 at a rate of 1 L/min. Water flows from tank 1 to tank 3 at a rate of 3 L/min, and from tank 3 to tank 1 at a rate of 3 L/min. Water flows from tank 2 to tank 3 at a rate of 4 L/min, and from tank 3 to tank 2 at a rate of 4 L/min. Water also flows from tank 2 out of the system at a rate of 1 L/min.

   Note that with this setup, the volume in each tank is constant. Assume that the tanks are full of water, that each tank starts with a concentration of 0 g/L of salt, and that the water flowing into tank 1 from an outside source has a concentration of 0.5 g/L. Let $Q_1(t)$, $Q_2(t)$ and $Q_3(t)$ denote the amount of salt in tanks 1, 2 and 3 respectively. Write down a system of differential equations and initial value problem describing the quantity $\mathbf{Q}(t) = \begin{pmatrix} Q_1(t) \\ Q_2(t) \\ Q_3(t) \end{pmatrix}$, of the form

$$\mathbf{Q}' = A\mathbf{Q} + \mathbf{B},\ \mathbf{Q}(0) = \mathbf{C},$$

   where A is a square matrix, $\mathbf{B}$ is a vector function, and $\mathbf{C}$ is a constant vector.

   Solution:

$$\mathbf{Q}' = \frac{1}{1000}\begin{pmatrix} -5 & 1 & 3 \\ 2 & -6 & 4 \\ 3 & 4 & -7 \end{pmatrix}\mathbf{Q} + \begin{pmatrix} 0.5 \\ 0 \\ 0 \end{pmatrix},\ \mathbf{Q}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

5. Suppose A is a square matrix. Recall that

$$e^A := I + \sum_{n=1}^{\infty} \frac{A^n}{n!}$$

Prove the following equality:

$$\det(e^A) = e^{\mathrm{tr}(A)}.$$

Solution: This is a simple consequence of the facts that the determinant of a matrix is the product of its eigenvalues with multiplicity and the trace is the sum with multiplicity. Further, if λ is an eigenvalue of A then e^λ is an eigenvalue of e^A with the same multiplicity. Thus

$$\det(e^A) = \prod_i e^{\lambda_i} = e^{\sum_i \lambda_i} = e^{\mathrm{tr} A}.$$