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Math 309 Final

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Instructions: You may use a calculator for this exam. Please turn off all cell phones and pagers. You must show all work. Wherever a general solution is required, the solution must be in explicit form.

(20) 1. Solve the following initial value problem for $\mathbf{x}$.

$$\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \mathbf{x}; \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Since the matrix is upper triangular, the diagonal entries are the eigenvalues with multiplicity, so $\chi_A(x) = (x-2)^2$. We first compute $A - 2I = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$. The kernel of this matrix is spanned by $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is up to scaling the unique eigenvector. Thus we must find a generalized eigenvector η such that $(A-2I)\eta = \xi$, i.e. $\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. One solution is $\eta = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$. The general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 \xi e^{2t} + c_2(\xi t + \eta) e^{2t} \\ &= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} \right) e^{2t}. \end{aligned}$$

Plugging in $t = 0$ gives

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2/3 \end{pmatrix}.$$

Thus $c_1 = 1$, $c_2 = 6$, and the solution is

$$\mathbf{x}(t) = \begin{pmatrix} 1 + 6t \\ 2 \end{pmatrix} e^{2t}.$$

(20) 2. Find the general solution to the following system of differential equations.

$$\mathbf{x}' = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ 0 \\ e^{4t} \end{pmatrix}$$

Since $\det(A - \lambda I) = (3 - \lambda)\det \begin{pmatrix} -\lambda & 2 \\ -2 & -\lambda \end{pmatrix} = (3-\lambda)(\lambda^2 + 4)$, the eigenvalues are $3, \pm 2i$. If we compute eigenvectors, we get

$$\xi_3 = \begin{pmatrix} 2 \\ 3 \\ 13 \end{pmatrix}$$

$$\xi_{2i} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

$$\xi_{-2i} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}.$$

The general complex solution is

$$\mathbf{x}_h(t) = c_1 \xi_3 e^{3t} + c_2 \xi_{2i} e^{2it} + c_3 \xi_{-2i} e^{-2it},$$

and the general real solution is

$$\begin{aligned} \mathbf{x}_h(t) &= c_1 \begin{pmatrix} 2 \\ 3 \\ 13 \end{pmatrix} e^{3t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sin(2t) \right] + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cos(2t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sin(2t) \right] \\ &= c_1 \begin{pmatrix} 2 \\ 3 \\ 13 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} \cos(2t) \\ -\sin(2t) \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} \sin(2t) \\ \cos(2t) \\ 0 \end{pmatrix}. \end{aligned}$$

To find a particular solution, assume it is of the form $\alpha e^t + \beta e^{4t}$. Then one finds that $\alpha = \begin{pmatrix} 1/5 \\ -2/5 \\ 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1/10 \\ 1/5 \\ 1 \end{pmatrix}$. Thus the final solution is

$$c_1 \begin{pmatrix} 2 \\ 3 \\ 13 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} \cos(2t) \\ -\sin(2t) \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} \sin(2t) \\ \cos(2t) \\ 0 \end{pmatrix} + \begin{pmatrix} 1/5 \\ -2/5 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 1/10 \\ 1/5 \\ 1 \end{pmatrix} e^{4t}.$$

(20) 3. With

$$A = \begin{pmatrix} 5 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix},$$

compute e^A as a product of matrices, and their inverses. If you do not wish to compute the inverse of a matrix, you can explain how you would do it for full credit.

First observe that the matrix is upper triangular, hence its diagonal entries are its eigenvalues; since they are all distinct, this matrix is diagonalizable. Compute its eigenvectors:

$$\xi_2 = \begin{pmatrix} 7/3 \\ -4 \\ 1 \end{pmatrix}$$

$$\xi_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\xi_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Arranging them in a matrix

$$T = \begin{pmatrix} 7/3 & -1 & 1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which represents a change of basis, such that the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is taken the the eigenvectors ξ_2, ξ_3, ξ_5. This transforms the matrix A into a diagonal matrix, $D := T^{-1}AT = \operatorname{diag}(2, 3, 5)$. Then we have

$$\begin{aligned} e^A &= e^{TDT^{-1}} \\ &= \sum_{n=0}^{\infty} \frac{(TDT^{-1})^n}{n!} = T \left(\sum_{n=0}^{\infty} \frac{D^n}{n!} \right) T^{-1} = T e^D T^{-1} \\ &= T \begin{pmatrix} e^2 & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^5 \end{pmatrix} T^{-1}. \end{aligned}$$

Inverting the matrix is accomplished by augmenting T with the identity I and row reducing T to the identity. Then the nonpivot columns will form T^{-1}:

$$\left[\, T \mid I \,\right] \xrightarrow{\text{row reduce}} \left[\, I \mid T^{-1} \,\right].$$

(25) 4. Compute a sine Fourier series for the function f on the interval $[0, \pi]$ where $f(x) = 1$ for all x.

Clarification: To give a sine series, the function $f(x) = 1$ is extended to an odd function g where

$$g(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 0 & x = 0 \\ 1 & 0 < x \le \pi. \end{cases}$$

Solution: The usual formula for sine series is used:

$$\sum_{n=1}^{\infty} a_n \sin(nx),$$

where

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\ dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sin(nx)\ dx \\ &= -\frac{2}{\pi n} \cos(nx) \Big|_0^{\pi} \\ &= -\frac{2}{\pi n} \Big((-1)^n - 1 \Big) \\ &= \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi n} & n \text{ odd.} \end{cases} \end{aligned}$$

Thus the sine series for f is

$$f(x) = 1 = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{\pi n} \sin(nx), \quad 0 \le x \le \pi.$$

(25) 5. Solve the following boundary value problem for a function $u(x,t)$ defined for $0 \leq x \leq \pi/2$ and $t \geq 0$ which satisfies

- $u_{tt} = u_{xx}, 0 < x < \pi/2, t > 0,$
- $u_t(x,0) = 0, 0 \leq x \leq \pi/2,$
- $u(0,t) = 0, t \geq 0,$
- $u_x(\pi/2, t) = 0, t \geq 0,$ and
- $u(x,0) = 1, 0 < x < \pi/2.$

We begin by considering product solutions, of the form $u(x,t) = X(x)T(t)$. If this is the case, then the differential equation $u_{tt} = u_{xx}$ translates to $X(x)T''(t) = X''(x)T(t)$. Since we are looking for nonzero solutions, we can assume that $T(t)$ and $X(x)$ are not always zero, and when this is the case, we have

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}.$$

Fixing x shows that the right side does not depend on t and fixing t shows that the left side does not depend on x. Therefore each side is equal to a common constant λ:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda, \quad \text{i.e.} \quad X''(x) = \lambda X(x) \text{ and } T''(t) = \lambda T(t).$$

We then translate the homogeneous boundary conditions to conditions on X and T, in particular,

$$\begin{aligned} u_t(x,0) = 0 &\Rightarrow T'(0) = 0 \\ u(0,t) = 0 &\Rightarrow X(0) = 0, \text{ and} \\ u_x(\pi/2,t) = 0 &\Rightarrow X'(\pi/2) = 0. \end{aligned}$$

If $\lambda = 0$, then $X(x) = cx + d$, and $X'(\pi/2) = 0$ implies $c = 0$, while $X(0) = 0$ implies $d = 0$. Thus to $\lambda = 0$ corresponds only the trivial solution $X(x)T(t) = 0$. Suppose $\lambda \neq 0$. Since the general solution for $T''(t) = \lambda T(t)$ is

$$T(t) = c_1 \cos\left(\sqrt{-\lambda} t\right) + c_2 \sin\left(\sqrt{-\lambda} t\right),$$

$T'(0) = 0$ implies that $c_2 = 0$. Thus, up to a constant,

$$T(t) = \cos\left(\sqrt{-\lambda} t\right).$$

Similarly, the general solution for $X''(x) = \lambda X(x)$ is

$$X(x) = c_1 \cos\left(\sqrt{-\lambda} x\right) + c_2 \sin\left(\sqrt{-\lambda} x\right),$$

and $X(0) = 0$ implies that $c_1 = 0$. Then $X'(x) = c_2\sqrt{-\lambda}\cos\left(\sqrt{-\lambda}x\right)$, and $X'(\pi/2) = 0$ gives $c_2\sqrt{-\lambda}\cos\left(\frac{\sqrt{-\lambda}\pi}{2}\right) = 0$, which implies that $\frac{\sqrt{-\lambda}\pi}{2} = \frac{n\pi}{2}$ for some odd n, in other words, $\lambda = -n^2$ for some odd n, and our solution is

$$\sin(nx)\cos(nt), n \text{ odd.}$$

Since we have only so far used homogeneous conditions, we may assume there is a solution in the form

$$u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} a_n \sin(nx)\cos(nt).$$

Further, since $u(x,0) = \sum_{n=1,3,5,\ldots}^{\infty} a_n \sin(nx) = 1, 0 < x < \pi/2$, we see by the previous problem that $a_n = \frac{4}{\pi n}$. Therefore

$$u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{\pi n}\sin(nx)\cos(nt)$$

is a solution.